Kepler orbit in the gravitational field of a Schwarzschild black hole and perihelion precession

Teorija gravitacije - projekt

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The first part of this article deals with the orbit of a body in the vicinity of a Schwarzschild black hole. In the second part, the perihelion precession of such an orbit will be discussed.

1 Kepler orbit in the gravitational field of a Schwarzschild black hole

For the study of the equations of motion in the gravitational field of a Schwarzschild black hole, we use the Schwarzschild coordinates t, r, θ, ϕ and the Schwarzschild metric

$$g_{\mu\nu} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & \frac{1}{1-2M/r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

where $M = \frac{GM'}{c^2}$ is the gravitational radius of a black hole with mass M'.
As has been shown in Andreja Gomboc's dissertation [1], the orbit of a body in the vicinity of a non-rotating, uncharged black hole is planar. We can choose $\theta = \frac{\pi}{2}$ [3] for the invariant orbit plane. The orbit equation, a relation between the variable r and the angle ϕ, is then given [1],[3] by

$$\left(\frac{du}{d\phi}\right)^2 = \left(2M\frac{E}{L}\right)^2 + 2H\left(\frac{2M}{L}\right)^2(1-u) - u^2(1-u), \tag{1}$$

where $u = \frac{2M}{r}$ and L denotes the angular momentum.
Therefore, to obtain the orbit equation, we have to solve the integral

$$\int d\phi = \pm\int \frac{du}{\sqrt{P(u)}},$$

where

$$P(u) = \left(2M\frac{E}{L}\right)^2 + 2H\left(\frac{2M}{L}\right)^2(1-u) - u^2(1-u)$$

is a third degree polynomial in u. As has been discussed in Andreja Gomboc's dissertation as well, the bound orbits (Type D orbits) correspond to the case with three real roots u_1, u_2 and u_3 in the range between 0 and 1. The smallest two roots, let's say u_3 and u_2 as in the dissertation, correspond to the perihelion and aphelion of the orbit respectively.
As we are interested in the bound orbits only, we consider values for u where $u_3 < u < u_2$. We thus have to solve the integral

$$\pm \int \frac{du}{\sqrt{P(u)}},$$

with three real roots $u_3 < u_2 < u_1$ of $P(u)$, and where $u_3 < u < u_2$. We can show [2] that the substitution $u = u_3 + (u_2 - u_3)\sin^2\chi$ transforms this integral into

$$\frac{2}{\sqrt{u_1 - u_3}} \int \frac{d\chi}{\sqrt{1 - k^2 \sin^2\chi}},$$

where the modulus k is given by

$$k^2 = \frac{u_2 - u_3}{u_1 - u_3}.$$

This elliptic integral, integrated from an initial value χ_i to χ, has the solution

$$\pm\frac{2}{\sqrt{u_1 - u_3}} \int \frac{d\chi}{\sqrt{1 - k^2 \sin^2\chi}} = \pm\frac{2}{\sqrt{u_1 - u_3}} \left(F(\chi, k) - F(\chi_i, k)\right),$$

where $F(\chi, k)$ is the elliptic integral of the first kind. We can express u in terms of ϕ, as shown in the dissertation, and finally the orbit equation is given by

$$u = u_2 - (u_2 - u_3)\mathrm{cn}^2\left(F(\chi_i, k) \pm \frac{(\phi - \phi_i)\sqrt{u_1 - u_3}}{2}, k\right).$$

The orbit equation is thus the same as for Type A orbits in the dissertation. The only difference to Type A orbits is the value of the energy: For bound orbits,

$$E^2 < 1.$$

For massive particles, $H = -\frac{1}{2}$. By defining the reduced angular momentum $\tilde{L}$ by

$$\tilde{L} = \frac{L}{2M},$$

we can draw different orbits for different values of $\tilde{L}$ and E. In all the following graphics, ϕ runs from 0 to 16π.

1.1 Orbits with energy $E = 0.98$

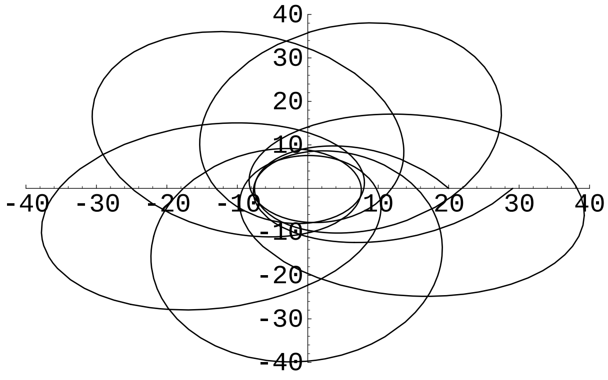

Abbildung 1: $E = 0.98, \tilde{L} = 2.1$

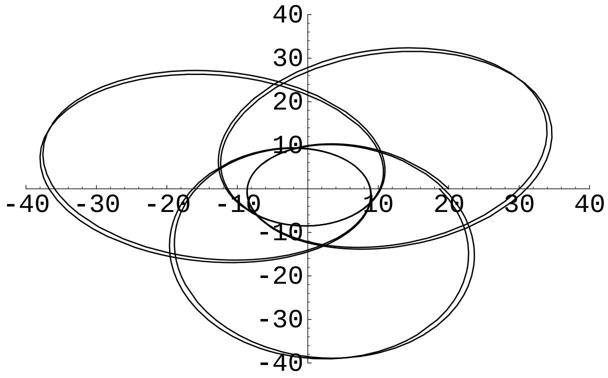

Abbildung 2: $E = 0.98, \tilde{L} = 2.16$

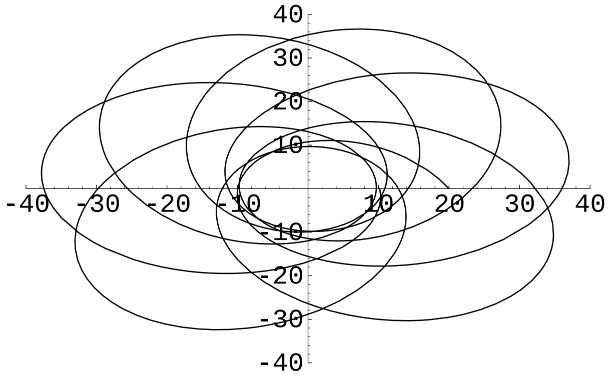

Abbildung 3: $E = 0.98, \tilde{L} = 2.22$

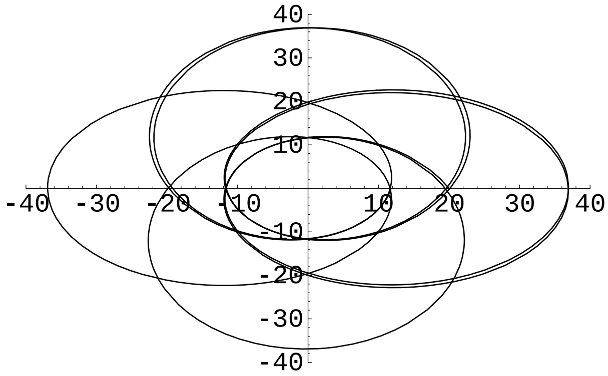

Abbildung 4: $E = 0.98, \tilde{L} = 2.29$

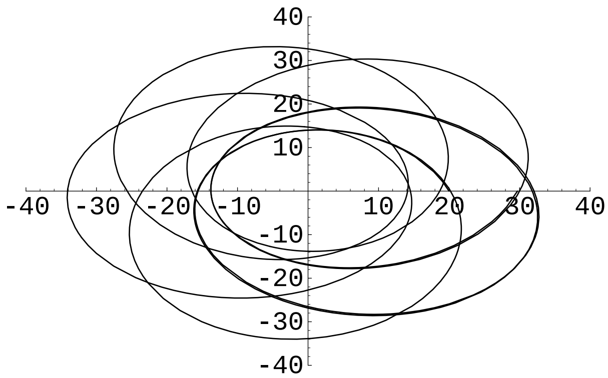

Abbildung 5: $E = 0.98, \tilde{L} = 2.42$

1.2 Orbits with energy $E = 0.97$

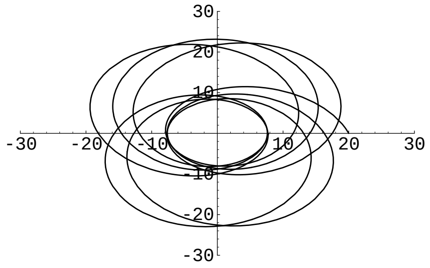

Abbildung 6: $E = 0.97, \tilde{L} = 2.01$

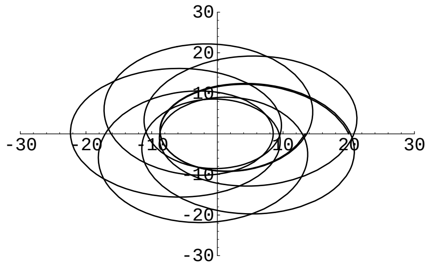

Abbildung 7: $E = 0.97, \tilde{L} = 2.05$

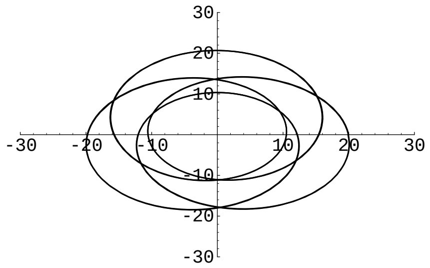

Abbildung 8: $E = 0.97, \tilde{L} = 2.12$

1.3 Orbits with reduced angular momentum $\tilde{L} = 2.1$

The following graphics show orbits for $\tilde{L} = 2.1$ and changing values of the energy.

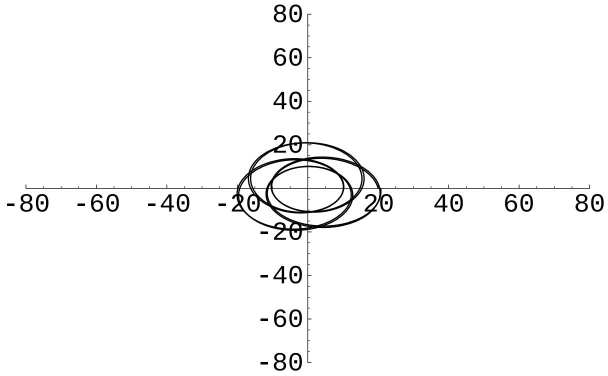

Abbildung 9: $E = 0.97, \tilde{L} = 2.1$

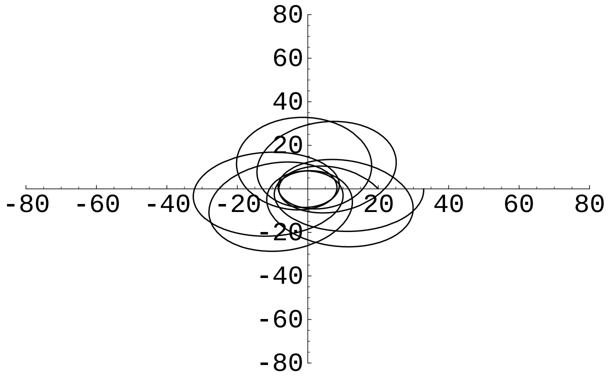

Abbildung 10: $E = 0.978, \tilde{L} = 2.1$

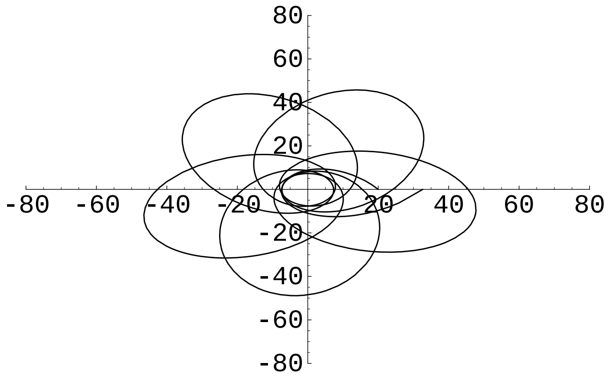

Abbildung 11: $E = 0.983, \tilde{L} = 2.1$

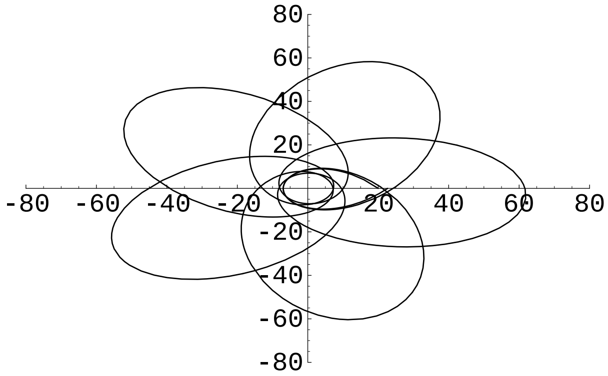

Abbildung 12: $E = 0.986, \tilde{L} = 2.1$

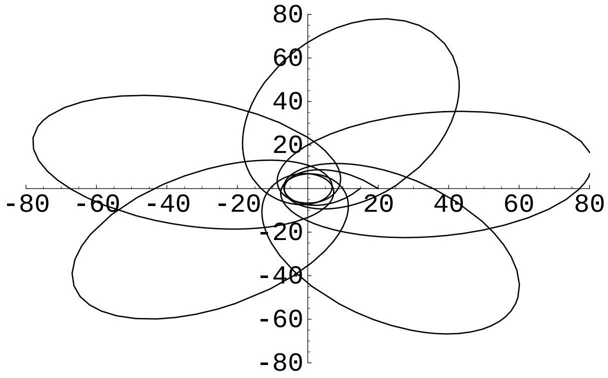

Abbildung 13: $E = 0.988, \tilde{L} = 2.1$

2 Perihelion precession

We would now like to calculate the precession of the perihelion. As the orbit is nearly elliptical, it makes sense to introduce the substitution

$$\frac{u}{2M} = \frac{1}{r} = \frac{1}{l}(1 + e\cos\chi)$$

as done in [3], where l and e are the latus rectum and the eccentricity of the ellipse respectively. The angle χ is chosen so that $\chi = 0$ at perihelion and $\chi = \pi$ at aphelion.
For massive particles with $H = -\frac{1}{2}$, the orbit equation (1) is

$$\left(\frac{du}{d\phi}\right)^2 = (2M\frac{E}{L})^2 - \left(\frac{2M}{L}\right)^2 (1-u) - u^2(1-u).$$

After some calculations [3], this equation simplifies with the above substitution to

$$\left(\frac{d\chi}{d\phi}\right)^2 = 1 - 2\mu(3 + e\cos\chi), \tag{2}$$

where $\mu = \frac{M}{l}$.
The parameter μ is usually very small, as it essentially represents the ratio of the Schwarzschild radius and the latus rectum of a planet orbit. Therefore we can expand (2) to the first order in μ:

$$\begin{aligned} d\phi &= \pm\frac{1}{\sqrt{1-2\mu(3+e\cos\chi}}d\chi \\ &\approx \pm[1+\mu(3+e\cos\chi)]d\chi. \end{aligned} \tag{3}$$

We are interested in the change of the angle ϕ during a complete ellipse revolution, where χ changes by 2π. Finally, the perihelion precession is then given by

$$\Delta\phi = \int_0^{2\pi}[1+\mu(3+e\cos\chi)]d\chi - 2\pi = 2(1+3\mu)\pi - 2\pi = 6\pi\frac{M}{l} = 6\pi\frac{M}{a(1-e^2)}.$$

In the following table the numerical values are given for Mercury, Venus and the Earth.

Planet	a	e	$\Delta\phi$	T
Mercury	$5.7909 \cdot 10^{10}$ m	0.2056	$5.01 \cdot 10^{-7}$	87.9 days
Venus	$2.164 \cdot 10^{11}$ m	0.0067	$1.29 \cdot 10^{-7}$	224.7 days
Earth	$2.9919 \cdot 10^{11}$ m	0.0167	$9.30 \cdot 10^{-6}$	365.26 days

For Mercury we get a perihelion precession of about 43 seconds of arc per century, which coincides with the observed value.

2.1 The Laplace-Runge-Lenz vector and the angular momentum

The classical Laplace-Runge-Lenz vector is defined by

$$\vec{A} = \vec{p} \times \vec{L} - GmM' \frac{\vec{r}}{r},$$

where $r = |\vec{r}|$. It is known that the Laplace-Runge-Lenz vector points towards the perihelion, from the geometrical center of the ellipse to the central body (in this case the black hole). If we denote by χ the angle between the position and the Laplace-Runge-Lenz vector, we can write

$$\vec{A} \cdot \vec{r} = Ar\cos\chi = (\vec{p} \times \vec{L}) \cdot \vec{r} - GmM'r = \vec{L}(\vec{r} \times \vec{p}) - GmM'r = L^2 - GmM'r.$$

From this we conclude

$$r = \frac{L^2}{GmM' + A\cos\chi} = \frac{\frac{L^2}{GmM'}}{1 + \frac{A}{GmM'}\cos\chi}. \tag{4}$$

In polar coordinates, where the origin is one focus of the ellipse,

$$r = \frac{a(1-e^2)}{1 + e\cos\chi}. \tag{5}$$

Comparing expression (5) with (4), we get two relations between the ellipse parameters, the Laplace-Runge-Lenz vector and the angular momentum:

$$a(1-e^2) = \frac{L^2}{GmM'}$$

and

$$e = \frac{A}{GmM'}.$$

These results are valid classically. I doubt that they can be used in Schwarzschild space-time as well, because a relativistic version of the Laplace-Runge-Lenz vector would have to be defined first.
But in an almost flat space-time, we can probably use the above classical results, and insert them into the equation for $\Delta\phi$. The advance in the perihelion per revolution would be in that case given by

$$\Delta\phi = 6\pi \frac{M}{\frac{L^2}{GmM'}} = 6\pi \frac{MM'Gm}{L^2}.$$

3 References

[1] A. Gomboc. *Rapid Luminosity Changes due to Interaction with a Black Hole*. Dissertation, mentor A. Čadež. University of Ljubljana, 2001

[2] H. Hancock. *Elliptic Integrals*. Dover Publications, 1958

[3] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford University Press Inc., 1992